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Elementary Applications

Velocity of escape from the earth:

A particle is projected from the earth with ~~the~~ a velocity v_0 such that $v_0 \geq \sqrt{2gR}$ ~~so th~~ (R is the radius of earth g is the accelerat-
-ion of gravity at the surface of earth). ~~So that the particle~~ will escape from the earth. Hence the minimum, such velocity of projection is $V_e = \sqrt{2gR}$ is called velocity of escape.

The radius of the earth is approximately. $R = 3960$ miles and the acceleration of gravity at the surfaces of the earth is approximately $g = 6.09 \times 10^{-3}$ miles/sec.

$$v_e = \sqrt{2gR} = \sqrt{2 \times 6.09 \times 10^{-3} \times 3960}$$
$$= 6.95 \text{ miles/sec.}$$

Exercise:

1. The radius of the moon is approximately (roughly) 1080 miles ~~The~~ the acceleration of gravity at the surface of the moon is about $0.165g$, where g is the

acceleration of gravity at the surface of the earth. ~~Determine the~~ Determine the velocity of escape for the moon.

= The acceleration of gravity at the surface of the moon is, $g' = 0.165\,g$

$= 0.165 \times 6.09 \times 10^{-3}$

$= \underline{1.00485 \times 10^{-3}}$ miles/sec

$\therefore v_e = \sqrt{2g'R}$

$= \sqrt{2 \times 1.00485 \times 10^{-3} \times 1080}$

$= \underline{1.5}$ miles/sec

Newton's law of cooling:

The temperature of the object at various tim[e] can be obtained by using Newton's law of ~~coll~~ ~~-ng~~ cooling, which states that, the temperature of a body changes at a rate which is proportional to the difference in temperature between the outsid[e] medium and the body itself.

(we shall assume ~~#~~ here that constant of proportionality is the same whether the temper-ture is increasing or decreasing)

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~~Example: Suppose, for instance, that a thermometer, which has been~~

Example: Suppose a thermometer reading 70°F inside a house, when it is placed outside where the air temperature is 10°F, 3 minutes later it is found that the thermometer's reading 25°F we wish to predict the thermometer reading at various later times.

Let u°F represent the temperature of the thermometer at time t minutes.

According to ~~It is given that by~~ Newton's law of cooling we have, $\frac{du}{dt} \propto (u-10)$

Farad -F

$$\frac{du}{dt} = -k(u-10),$$

where k is the proportionality constant which is taken as negetive because the thermometer reading is decreasing. (70→25)

$$\frac{du}{dt} = -k(u-10)$$

$$\frac{du}{u-10} = -k\,dt.$$

Integrating, $\ln(u-10) = -kt+C,$

$(u-10) = e^{-kt+c_1}$

$(u-10) = e^{-kt} \cdot c$ — ① where $c = e^{c_1}$

Initial conditions are,

when $t=0$, $u=70$

$t=3$, $u=25$

Put $t=0$, $u=70$ in eqn ①.

$(70-10) = e^{0} \cdot c$

$\Rightarrow \underline{c = 60}$ $(\because e^{0}=1)$

$\therefore$ eqn ① becomes,

$(u-10) = e^{-kt} \cdot 60$

$u = e^{-kt} \cdot 60 + 10$

$u = 10 + e^{-kt} \cdot 60$ — ②

Put $t=3$, $u=25$ in eqn ②

$25 = 10 + e^{-k \cdot 3} \cdot 60$

$25 = 10 + e^{-3k} \cdot 60$

$15 = e^{-3k} \cdot 60$

$\frac{1}{4} = e^{-3k}$

$\ln\left(\frac{1}{4}\right) = -3k$

$k = -\frac{1}{3} \ln\left(\frac{1}{4}\right)$

$$K = \frac{1}{3} \ln\left(\frac{1}{4}\right)^{-1}$$

$$k = \frac{1}{3} \ln 4$$

substitute for k in eqn ⓐ

$$u = 10 + e^{-\left(\frac{1}{3}\ln 4\right).t} \cdot 60$$

$$u = 10 + 60 \cdot e^{\left(-\frac{1}{3}\ln 4\right).t}$$

(or) $u = 10 + 60\, e^{(-0.46t)}$ (since, $\ln 4 = 1.39$)

Simple chemical conversion:

If a substance A is being converted into another substance by some chemical reaction, the rate of change of the amount x of unconverted substance with respect to time is proportional to x.

Let the amount of unconverted substance at $t=0$ be x_0.

i.e. $x = x_0$ when $t = 0$.

Then the amount x at any time $t > 0$ is determined by the differential equation

$\frac{dx}{dt} = -kx$, where k is proportionality constant.

which is considered as negative because the amount x is decreasing as t increases

Solving, $\frac{dx}{dt} = -kx$

$$\frac{dx}{x} = -k\,dt$$

Integrating,

$$\ln x = -kt + c_1$$

$$x = e^{-kt + c_1}$$

$$x = e^{-kt} \cdot e^{c_1}$$

$$x = c e^{-kt} \quad -(1) \quad \text{where } c = e^{c_1}$$

but $x = x_0$ when $t = 0$.

$$x_0 = c e^{0}$$

$$x_0 = c$$

eqn (1) becomes $x = x_0 e^{-kt}$ —(2)

Let us consider another condition, to determine k. Suppose at the end ~~to~~ ~~k~~ of half a minute. i.e. when $t = 30$ sec., $\frac{2}{3}$ of the original amount x_0 has already been converted. Let us determine how ~~much~~ much unconverted substance remains at $t = 60$ s. when $\frac{2}{3}$ of original amount x_0 has been converted, $\frac{1}{3}$ ~~remo~~ of original amount remains unconverted.

Hence $x = \frac{1}{3}x_0$ when $t = 30$ x - unconverted substance 57

Put ∴ eqn ② becomes,

$$\frac{1}{3}x_0 = x_0 e^{-30k}$$

$$\frac{1}{3} = e^{-30k}$$

$$\ln\frac{1}{3} = -30k$$

$$k = \frac{1}{30}\left(-\ln\frac{1}{3}\right)$$

$$k = \frac{1}{30}\ln 3$$

Substitute in eqn ②,

$$\Rightarrow x = x_0 e^{\left(-\frac{1}{30}\ln 3\right)t} \quad \text{--- ③}$$

To find the amount of the unconverted substance when $t=60$

$t=60$ in eqn ③

$$x = x_0 e^{\left(-\frac{1}{30}\ln 3\right).60}$$

$$= x_0 e^{-2\ln 3}$$

$$= x_0 e^{\ln(3)^{-2}}$$

$$= x_0 e^{\ln\frac{1}{9}}$$

$$x = x_0 \frac{1}{9}$$

$$x = \frac{1}{9}.x_0$$